Primer on Inventory Management – Solutions

Service Levels

Solution to Exercise 1:

i. We know that

$$\beta \approx 1 - \frac{L\left(\frac{S^\star - \mu_{LT+1}}{\sigma_{LT+1}}\right) \cdot \sigma_{LT+1}}{\mu} \tag{46}$$

and subsequently

$$S^\star \approx \mu_{LT+1} + L^{-1}\left(\frac{(1-\beta)\cdot\mu}{\sigma_{LT+1}}\right) \cdot \sigma_{LT+1}. \tag{47}$$

In our case

$$S^\star = 100 \cdot (34+1) + L^{-1}\left(\frac{(1-0.99)\cdot 100}{\sqrt{34+1}\cdot 30}\right) \cdot \sqrt{34+1} \cdot 30 \tag{48}$$

$$= 3500 + L^{-1}(0.0056) \cdot 177.48 \tag{49}$$

$$= 3500 + 2.15 \cdot 177.48 \tag{50}$$

$$= 3881.58 \tag{51}$$

ii. In the standard cost model, $S^\star$ is computed from

$$S^\star = \mu_{LT+1} + z^\star \cdot \sigma_{LT+1}, \tag{52}$$

where

$$z^\star = F^{-1}\left(\frac{p}{p+h}\right). \tag{53}$$

Obviously, the cost models yields the same $S^\star$ as the β-Service-Level model, if

$$z^\star = L^{-1}\left(\frac{(1-0.99)\cdot 100}{\sqrt{34+1}\cdot 30}\right) = 2.15. \tag{54}$$

Hence, we obtain

$$\frac{p}{p+h} = F(2.15) = 0.9842 \tag{55}$$

and finally

$$p = \frac{0.9842 \cdot h}{1-0.9842} = \frac{0.9842 \cdot 1}{0.0158} = 62.29. \tag{56}$$

iii. Recall that the β-Service-Level is defined as the fraction of demand that is satisfied in a period. This is equal to the fraction of demand that is *not* backordered. Since $\beta = 0.99$, we require that on average 1% of the average period demand of 100 units is backordered. Therefore, expected backorders per period will be 1 unit.

For the sake of completeness, we also present the computation: The expected average backorder level can be comnputed as

$$E\left[Y^{LT+1} - S\right]^{+} = \sum_{y=S}^{\infty} (y-S) \cdot f_{LT+1}(y)\, dy \tag{57}$$

$$= L\left(\frac{S^{\star} - \mu_{LT+1}}{\sigma_{LT+1}}\right) \cdot \sigma_{LT+1} \tag{58}$$

$$= L(2.15) \cdot 177.48 \tag{59}$$

$$= 0.0056 \cdot 177.48 \tag{60}$$

$$\approx 1. \tag{61}$$

Observe, that the Loss function term has already been computed above.

Solution to Exercise 2:

Subsequently, we derive expressions for the expected shortage at the beginning and at the end of a cycle. Let us tag an arbitrary replenishment order. The inventory position just prior to the placing of the tagged order is $S - Y_R$, where Y_R is the cumulative demand since the previous review. Any stock that was on order prior to placing our tagged order has already arrived when the tagged order comes in. Thus, the net stock before the arrival of the order is $S - Y_R - Y_{LT}$, where Y_{LT} is the cumulative demand during the replenishment lead time. Since Y_R and Y_{LT} are independent, we can write $S - Y_{LT+R}$. This means, that the net stock at the end of a cycle – just before an order arrives – is distributed as $S - Y_{LT+R}$ and we have our familiar expression:

$$E\left[\text{Backorders at the end of a cylce}\right] = E\left[Y^{LT+R} - S\right]^{+} = \sum_{y=S}^{\infty} (y-S) \cdot f_{LT+R}(y)\, dy \tag{62}$$

Further, since the size of any replenishment order is distributed as the demand during the replenishment interval, i.e. Y_R, we have that the net stock at the beginning of a cycle – just after the arrival of a replenishment order – is dsitrubuted as $S - Y_{LT+R} + Y_R = S - Y_{LT}$ and we have:

$$E\left[\text{Backorders at the beginning of a cylce}\right] = E\left[Y^{LT} - S\right]^{+} = \sum_{y=S}^{\infty} (y - S) \cdot f_{LT}(y)\, dy \tag{63}$$

The expected demand in one cycle is μ_R. Subsequently, we obtain

$$\begin{aligned}
\beta &= 1 - \frac{E\left[Y^{LT+R} - S\right]^{+} - E\left[Y^{LT} - S\right]^{+}}{\mu_R} && (64)\\
&= 1 - \frac{1}{\mu_R}\left[\sum_{y=S}^{\infty} (y - S) \cdot f_{LT+R}(y)\, dy - \sum_{y=S}^{\infty} (y - S) \cdot f_{LT}(y)\, dy\right] && (65)\\
&= 1 - \frac{1}{\mu_R}\left[\sigma_{LT+R} \cdot L\left(\frac{S - \mu_{LT+R}}{\sigma_{LT+R}}\right) - \sigma_{LT} \cdot L\left(\frac{S - \mu_{LT}}{\sigma_{LT}}\right)\right]. && (66)
\end{aligned}$$